

Orbital Functional Geometry II

Discrete Spectrum, Eigenfunctions, and the Hilbert Structure of $\mathcal{O}$

Preprint

Judicael Brindel

Independent Researcher, Cotonou, Benin

ORCID: 0009-0007-4590-9874

March 2026

Abstract

We continue the development of the Orbital Space $\mathcal{O}$, introduced in *Orbital Functional Geometry* (2026). The four open directions of that paper — discrete spectrum, full consequences of the Hausdorff metric, classification of orbital flows, and two-level topology — are here resolved.

The discrete spectrum of the orbital Laplacian $\Delta_{\mathcal{O}}$ is determined: under the periodicity constraint, the eigenvalue equation reduces to a forced harmonic oscillator on $[0, 2\pi]$. Resonance at mode n forces $\lambda_n = b$, producing a discrete spectrum indexed by the position parameter b . The quantisation condition $a^2 r^2 = 2/n^2$ links the orbital radius to the mode number, giving a sequence of degenerate levels $r_n = \sqrt{b}/n$ — an atomic structure intrinsic to $\mathcal{O}$.

The Hausdorff metric on this structure yields inter-level distances $d_H(C_n, C_{n+1}) = \sqrt{b}/n(n+1)$, a Cauchy sequence converging to the centre x_0 . This establishes that $\mathcal{O}$ is *not complete*: its Hausdorff completion $\bar{\mathcal{O}}$ adds precisely the roots of P as boundary points. The roots of P are simultaneously the structural centres of $\mathcal{O}$, the fixed points of rotation, and the boundary of its completion.

The three orbital flows are fully classified: the b -flow acts differentially on levels, the a -flow acts as a homothety of the atomic structure, and the x_0 -flow preserves all radii while deforming orbital shapes. The ratios $r_n/r_m = m/n$ are absolute invariants of $\mathcal{O}$, independent of all parameters and of f . This harmonic structure was not constructed — it emerged from the classification of flows.

The spectral structure of $\mathcal{O}$ is now complete:

$$\text{Spec}(\Delta_{\mathcal{O}}) = \underbrace{\left\{ \frac{2}{a^2} \right\}}_{\text{continuous, in } a} \cup \underbrace{\{b\}}_{\text{discrete, in } b}$$

The two spectral parts live in orthogonal parameters. Their intersection $2/a^2 = b$ defines a spectral degeneracy condition $a^2 b = 2$.

The complete eigenfunctions at degenerate level n are $f(\theta) = e^{R \cos(n\theta - \phi)}$ — exponentials of harmonic oscillations, parametrised by amplitude R and phase $\phi \in [0, 2\pi)$. Their Fourier expansion involves the modified Bessel functions $I_k(R)$, which appear without being imposed.

These eigenfunctions are orthogonal in the orbital Hilbert space $H_{\mathcal{O}}$, defined by the logarithmic inner product $\langle f, g \rangle_{\mathcal{O}} = \frac{1}{2\pi} \int_0^{2\pi} \ln f \cdot \ln g d\theta$. Every admissible f expands as a product of eigenfunctions, one per level. $\Delta_{\mathcal{O}}$ is self-adjoint on $H_{\mathcal{O}}$.

Each degenerate level $n \geq 1$ has multiplicity 2 (cosine and sine directions); level $n = 0$ has multiplicity 1. A mutual exclusion principle holds: for fixed (a, b, r) , at most one level n can be simultaneously degenerate. The invariant $a_n r_n = \sqrt{2}$ is constant across the entire tower of degenerate levels, independent of n , b , and f .

Contents

1	The Discrete Spectrum: Setup	5
2	Quantisation of the Orbital Radius	5
3	Resonance and the Discrete Eigenvalue	5
4	The Atomic Structure of $\mathcal{O}$	6
5	The Complete Spectrum of $\Delta_{\mathcal{O}}$	7
6	Hausdorff Geometry of the Atomic Structure	7
7	Classification of Orbital Flows	8
8	Absolute Invariants of $\mathcal{O}$	9
9	Two-Level Topology: Complete Characterisation	10
10	Complete Eigenfunctions	10
11	The Orbital Hilbert Space $H_{\mathcal{O}}$	11
12	Degeneracy Multiplicity and the Exclusion Principle	11
13	The Tower of Degenerate Levels	12
14	Asymptotic Behaviour of Eigenfunctions	12
15	The Spectral Measure of $\Delta_{\mathcal{O}}$	13

Introduction

The first paper established $\mathcal{O}$ from a single equation and left four directions open. This paper resolves them.

The method is the same: follow the mathematics without imposing structure. What emerged in the first paper — flat intrinsic geometry, charged extrinsic curvature, the dihedral symmetry group, the encoding theorem — emerged from the equation itself.

What emerges here is more surprising. The discrete spectrum produces an atomic structure: levels indexed by integers, with radii forming a harmonic series. The Hausdorff metric applied to these levels reveals that $\mathcal{O}$ is incomplete, and that its boundary is exactly the set of roots of P — the same roots that founded the space. The classification of flows reveals absolute invariants that depend on nothing.

None of this was sought. It followed from asking: what does the periodicity constraint actually impose?

1 The Discrete Spectrum: Setup

The orbital Laplacian, defined in the first paper, is:

$$\Delta_{\mathcal{O}} = \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} + r^2 \frac{\partial^2}{\partial x_0^2}$$

Its continuous spectrum is $\lambda = 2/a^2$, with eigenfunctions $f(x) = e^{\alpha x + \beta}$.

For the discrete spectrum, we impose the periodicity constraint: the orbit $z(\theta)$ must satisfy $z(\theta + 2\pi) = z(\theta)$, which requires $N = 0$ — no zeros of f inside the orbit.

Setting $u = \ln f$, the eigenvalue equation $\Delta_{\mathcal{O}} z = \lambda z$ becomes:

$$u'' + \frac{2}{a^2 r^2} u = \frac{(\lambda - b) P(x)}{r^2}$$

where $x = x_0 + r e^{i\theta}$ and differentiation is with respect to θ .

Setting $k^2 = 2/(a^2 r^2)$, this is:

$$u'' + k^2 u = \frac{(\lambda - b) P(x)}{r^2}$$

Remark 1 (Structure of the equation). *The left side is a harmonic oscillator in u . The right side is a forcing by $P(x)$ evaluated on the orbit circle. The nature of the discrete spectrum depends entirely on the Fourier modes of P on the circle.*

2 Quantisation of the Orbital Radius

The periodicity condition $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ requires the homogeneous solution $A \cos(k\theta) + B \sin(k\theta)$ to be periodic on $[0, 2\pi]$.

Lemma 1 (Quantisation condition). *The periodicity of the homogeneous solution requires:*

$$k = n \in \mathbb{Z}^* \implies \frac{2}{a^2 r^2} = n^2 \implies \boxed{a^2 r^2 = \frac{2}{n^2}}$$

The product ar is not free: it is constrained by the integer n . Only orbits satisfying this condition admit periodic eigenfunctions.

Remark 2 (Meaning of the quantisation). *The orbital radius r and the scale parameter a are not independent for periodic orbits. Each integer n selects a family of orbits with $r = \sqrt{2}/(|n| \cdot |a|)$. The integer n counts the number of oscillations of the eigenfunction around the orbit.*

3 Resonance and the Discrete Eigenvalue

On the orbit $x = x_0 + r e^{i\theta}$, the polynomial $P(x)$ expands in Fourier modes:

$$P(x) = \sum_{j=0}^m c_j r^j e^{ij\theta}$$

The forcing term becomes:

$$\frac{(\lambda - b) P(x)}{r^2} = (\lambda - b) \sum_{j=0}^m \frac{c_j}{r^{2-j}} e^{ij\theta}$$

For each Fourier mode $j \neq n$, the particular solution is:

$$u_p^{(j)} = \frac{(\lambda - b) c_j / r^{2-j}}{n^2 - j^2} e^{ij\theta}$$

This is well-defined and periodic.

Lemma 2 (Resonance condition). *When $j = n$, the forcing $e^{in\theta}$ matches the frequency of the oscillator. The particular solution acquires a secular term $\theta e^{in\theta}$, which is not periodic on $[0, 2\pi]$.*

Periodicity requires the resonant forcing to vanish:

$$(\lambda - b) c_n = 0$$

Theorem 1 (Discrete eigenvalue). *Let $P(x)$ have Fourier coefficient $c_n \neq 0$ on the orbit circle. Then the periodicity constraint forces:*

$$\boxed{\lambda_n = b}$$

The discrete eigenvalue of $\Delta_{\mathcal{O}}$ at mode n is the position parameter b .

The discrete spectrum is:

$$\text{Spec}_{\text{disc}}(\Delta_{\mathcal{O}}) = \{b\}$$

for all modes $n \leq \deg P$ with $c_n \neq 0$.

Remark 3 (Role of b). *In the first paper, b was the parameter of rigid translation: under $b \rightarrow -b$, all orbits shift uniformly by $2b/a$. Here b appears as the discrete eigenvalue of the orbital Laplacian. The same parameter governs both the global position of orbits and the discrete part of the spectrum. Translation and spectrum are the same object seen differently.*

4 The Atomic Structure of $\mathcal{O}$

The two conditions established so far:

1. Quantisation: $a^2 r^2 = 2/n^2$
2. Spectral degeneracy: $\lambda_{\text{cont}} = \lambda_{\text{disc}}$, i.e. $2/a^2 = b$, giving $a^2 b = 2$

Combining: $r^2 = 2/(n^2 a^2) = b/n^2$, so:

Definition 1 (Degenerate levels). *An orbit is degenerate if it satisfies both the quantisation condition and the spectral degeneracy condition. The degenerate levels are indexed by $n \in \mathbb{Z}^*$ with radii:*

$$\boxed{r_n = \frac{\sqrt{b}}{n}}$$

Lemma 3 (Atomic structure). *The degenerate levels $\{r_n\}_{n \geq 1}$ form a strictly decreasing sequence:*

$$r_1 > r_2 > r_3 > \cdots \rightarrow 0$$

All levels share the same centre x_0 and the same parameters (a, b) satisfying $a^2b = 2$.

This is the atomic structure of $\mathcal{O}$: discrete energy levels, indexed by integers, converging toward the centre.

Remark 4 (Emergence, not construction). *The atomic structure was not introduced as a hypothesis. It emerged from two independent conditions: the periodicity of eigenfunctions (quantisation) and the coincidence of continuous and discrete spectra (degeneracy). Neither condition was designed to produce levels.*

5 The Complete Spectrum of $\Delta_{\mathcal{O}}$

Theorem 2 (Complete spectral structure). *The spectrum of the orbital Laplacian is:*

$$\text{Spec}(\Delta_{\mathcal{O}}) = \underbrace{\left\{ \frac{2}{a^2} : a \in \mathbb{R}^* \right\}}_{\text{continuous, parameterised by } a} \cup \underbrace{\{b\}}_{\text{discrete, parameterised by } b}$$

The two parts have distinct properties:

- *Continuous spectrum: strictly positive, $\lambda = 2/a^2 > 0$, varies over $(0, +\infty)$ as a varies over $\mathbb{R}^*$*
- *Discrete spectrum: $\lambda = b$, a single value determined by the position parameter*
- *The two parameters a and b are independent in $\mathcal{O}$: the two spectral parts are orthogonal*

Definition 2 (Spectral degeneracy condition). *The continuous and discrete spectra coincide when:*

$$\frac{2}{a^2} = b \iff a^2b = 2$$

At this condition, a single eigenvalue belongs to both parts of the spectrum simultaneously. This is the spectral degeneracy condition of $\mathcal{O}$.

Remark 5 (The number 2). *The constant 2 in $a^2b = 2$ is the same constant that appears in the continuous eigenvalue $\lambda = 2/a^2$ and in the quantisation condition $a^2r^2 = 2/n^2$. It is not a free parameter: it is fixed by the second derivative $\partial^2 z / \partial a^2 = 2Q/a^3$, which itself comes from the orbital equation. The number 2 is intrinsic to $\mathcal{O}$.*

6 Hausdorff Geometry of the Atomic Structure

Lemma 4 (Inter-level distance). *Two degenerate orbits C_n and C_{n+1} at the same centre x_0 , same parameters (a, b) , with radii $r_n = \sqrt{b}/n$ and $r_{n+1} = \sqrt{b}/(n+1)$:*

$$d_H(C_n, C_{n+1}) = |r_n - r_{n+1}| = \sqrt{b} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{\sqrt{b}}{n(n+1)}$$

Lemma 5 (Cauchy property of the level sequence). *The sequence $(C_n)_{n \geq 1}$ is Cauchy in $(\mathcal{O}, d_H)$:*

$$d_H(C_n, C_{n+k}) \leq \sum_{j=n}^{n+k-1} \frac{\sqrt{b}}{j(j+1)} = \sqrt{b} \left(\frac{1}{n} - \frac{1}{n+k} \right) \rightarrow 0$$

as $n \rightarrow \infty$, uniformly in k .

The level radii satisfy $r_n \rightarrow 0$, so the orbits C_n converge toward the point x_0 in the Hausdorff metric.

Theorem 3 (Incompleteness of $\mathcal{O}$). *The Cauchy sequence (C_n) converges to the point x_0 — a degenerate orbit of radius zero. A point is not an admissible orbit in $\mathcal{O}$.*

Therefore: $\mathcal{O}$ is not complete under the Hausdorff metric. The limit of the level sequence lies outside $\mathcal{O}$.

Definition 3 (Hausdorff completion). *The Hausdorff completion $\bar{\mathcal{O}}$ of $\mathcal{O}$ is obtained by adjoining the limits of all Cauchy sequences of admissible orbits.*

The boundary of $\mathcal{O}$ in its completion is:

$$\partial\mathcal{O} = \{x_0 : P(x_0) = 0\}$$

the set of roots of P .

Theorem 4 (Roots of P as boundary). *The roots of P are simultaneously:*

1. *The structural centres of $\mathcal{O}$ (centres of rotation, fixed points)*
2. *The points around which orbits are defined (the parameter x_0)*
3. *The boundary points of $\mathcal{O}$ in its Hausdorff completion $\bar{\mathcal{O}}$*

P is the identity of the space in three independent senses: structural, parametric, and topological.

7 Classification of Orbital Flows

The three flows of $\mathcal{O}$ were identified in the first paper. Here we classify them completely using the atomic structure.

The b -flow on levels

Under $b \rightarrow b + \delta b$, the degenerate level radii shift:

$$r_n(b + \delta b) = \frac{\sqrt{b + \delta b}}{n} \approx r_n + \frac{\delta b}{2n\sqrt{b}}$$

Lemma 6 (Differential action of the b -flow). *The b -flow acts differentially on the atomic levels: the displacement of level n is $\delta r_n = \delta b / (2n\sqrt{b})$.*

Higher levels (large n , small r) displace less than lower levels. The b -flow is rigid for individual orbits but differential for the level structure.

The a -flow on levels

The quantisation condition $a^2 r^2 = 2/n^2$ gives $r_n(a) = \sqrt{2}/(na)$.

Lemma 7 (Homothety of the a -flow). *Under the a -flow, all degenerate levels scale uniformly:*

$$\frac{\partial r_n}{\partial a} = -\frac{r_n}{a}$$

The a -flow is a homothety of the atomic structure: all levels contract (or expand) at the same relative rate $1/a$. The ratios $r_n/r_m = m/n$ are preserved.

The x_0 -flow on levels

The degenerate level radii $r_n = \sqrt{b}/n$ depend only on b and n — not on x_0 .

Lemma 8 (Preservation by the x_0 -flow). *The x_0 -flow preserves all degenerate level radii. It deforms the shape of individual orbits (through the logarithmic derivative of f) while leaving the atomic structure invariant.*

Complete classification

Theorem 5 (Classification of orbital flows). *The three flows act on the atomic structure as follows:*

<i>Flow</i>	<i>Nature</i>	<i>Action on orbits</i>	<i>Action on levels</i>
b	<i>Rigid</i>	<i>Uniform translation</i>	<i>Differential shift</i>
a	<i>Flexible</i>	<i>Homothety</i>	<i>Uniform contraction</i>
x_0	<i>Functional</i>	<i>Deformation by f</i>	<i>Preservation</i>

Each flow has a distinct invariant:

- b -flow: inter-orbit distances preserved; level radii shift
- a -flow: ratios r_n/r_m preserved; absolute radii change
- x_0 -flow: all radii r_n preserved; orbital shapes change

8 Absolute Invariants of $\mathcal{O}$

Theorem 6 (Harmonic structure). *The ratios of degenerate level radii:*

$$\boxed{\frac{r_n}{r_m} = \frac{m}{n}}$$

are absolute invariants of $\mathcal{O}$: independent of a , b , x_0 , and f .

These ratios form the harmonic series $1, 1/2, 1/3, \dots$ relative to r_1 .

Remark 6 (Harmonic structure without harmonics). *The harmonic ratios m/n emerged from the quantisation of orbital periodicity. No harmonic analysis was performed. No Fourier series was imposed on the space. The structure is harmonic because periodicity, quantised by integers, produces integer ratios.*

Corollary 1 (Invariance under all deformations). *Any continuous deformation of $\mathcal{O}$ that preserves the orbital equation preserves the harmonic ratios. The ratios $r_n/r_m = m/n$ are topological invariants of $\mathcal{O}$.*

9 Two-Level Topology: Complete Characterisation

The two-level topology was identified in the first paper: P gives structural boundaries, f gives functional boundaries. The atomic structure adds a third level.

Theorem 7 (Three-level topology of $\mathcal{O}$). *The topology of $\mathcal{O}$ has three levels:*

1. **Structural level** (given by P): roots of P partition $\mathbb{R}$ (or $\mathbb{C}$) into zones; they are the centres of rotation and the boundary of $\bar{\mathcal{O}}$
2. **Functional level** (given by f): roots of f determine which zones are real or complex; they are the sources of extrinsic curvature and orbital deformation
3. **Spectral level** (given by (a, b)): the quantisation condition $a^2 r^2 = 2/n^2$ selects discrete radii within each zone; the harmonic ratios m/n organise the fine structure

The classification of $\mathcal{O}$ requires all three levels. Two spaces in the same geometric class (same P) and same nature pattern (same sign of f in each zone) may still differ in their spectral level (different $a^2 b$).

10 Complete Eigenfunctions

At the degenerate level n , with $\lambda = b$, the forcing term vanishes and the eigenvalue equation reduces to the homogeneous oscillator:

$$u'' + n^2 u = 0$$

The general periodic solution is:

$$u(\theta) = A \cos(n\theta) + B \sin(n\theta) = R \cos(n\theta - \phi)$$

where $R = \sqrt{A^2 + B^2} \geq 0$ is the amplitude and $\phi = \arctan(B/A) \in [0, 2\pi)$ is the phase.

Definition 4 (Orbital eigenfunction at level n). *The eigenfunction of $\Delta_{\mathcal{O}}$ at degenerate level n , with amplitude R and phase ϕ , is:*

$$f_n^{(R, \phi)}(\theta) = e^{R \cos(n\theta - \phi)}$$

This is an exponential of a harmonic oscillation — a von Mises distribution on the circle.

Lemma 9 (Fourier expansion via Bessel functions). *The orbital eigenfunction expands in Fourier modes as:*

$$e^{R \cos(n\theta - \phi)} = \sum_{k=-\infty}^{+\infty} I_k(R) e^{ik(n\theta - \phi)}$$

where $I_k(R)$ are the modified Bessel functions of the first kind. The Bessel functions appear as the coefficients of transition between the orbital level structure and the Fourier modes of f . They were not introduced — they emerged from the structure of the eigenfunctions.

11 The Orbital Hilbert Space $H_{\mathcal{O}}$

The eigenfunctions $f_n^{(R,\phi)}$ are not orthogonal in the standard L^2 inner product. Their natural orthogonality lives in a different space.

Definition 5 (Orbital Hilbert space). *Define:*

$$H_{\mathcal{O}} = \left\{ f > 0, \text{ analytic on } S^1 : \frac{1}{2\pi} \int_0^{2\pi} |\ln f(\theta)|^2 d\theta < \infty \right\}$$

with the orbital inner product:

$$\langle f, g \rangle_{\mathcal{O}} = \frac{1}{2\pi} \int_0^{2\pi} \ln f(\theta) \cdot \ln g(\theta) d\theta$$

Remark 7 (Why logarithmic). *The orbital equation defines z via $\ln f$. The natural inner product of $\mathcal{O}$ is therefore on $\ln f$, not on f itself. The space $H_{\mathcal{O}}$ is isometric to $L^2(S^1)$ via the map $f \mapsto \ln f$. The logarithmic structure of $\mathcal{O}$ determines the geometry of its functional space.*

Theorem 8 (Orbital Hilbert space and completeness). *$H_{\mathcal{O}}$ is a Hilbert space. The eigenfunctions $\{f_n^{(R,\phi)}\}$ form an orthogonal family in $H_{\mathcal{O}}$.*

Every admissible function f on an orbit expands as:

$$f(\theta) = \exp \left(\sum_{n=0}^{+\infty} R_n \cos(n\theta - \phi_n) \right) = \prod_{n=0}^{+\infty} f_n^{(R_n, \phi_n)}(\theta)$$

a product of eigenfunctions, one per level.

$\Delta_{\mathcal{O}}$ is self-adjoint on $H_{\mathcal{O}}$.

12 Degeneracy Multiplicity and the Exclusion Principle

Lemma 10 (Multiplicity at each level). *At degenerate level $n \geq 1$, the eigenspace of $\Delta_{\mathcal{O}}$ associated to $\lambda = b$ is two-dimensional, spanned by:*

$$f_n^c(\theta) = e^{A \cos(n\theta)}, \quad f_n^s(\theta) = e^{B \sin(n\theta)}$$

corresponding to the cosine and sine directions.

At level $n = 0$, the eigenspace is one-dimensional: $f_0(\theta) = e^A = \text{const.}$

The total discrete eigenspace has countably infinite dimension, with each level $n \geq 1$ contributing exactly 2.

Lemma 11 (Mutual exclusion of degenerate levels). *For fixed parameters (a, b, r) , the quantisation condition $a^2 r^2 = 2/n^2$ determines n uniquely. Two distinct levels $n \neq m$ cannot be simultaneously degenerate for the same orbit.*

At most one level n is degenerate for any given orbit.

Remark 8 (Structural analogy). *The mutual exclusion of degenerate levels parallels the Pauli exclusion principle: no two levels can occupy the same orbital state simultaneously. This is not an analogy imposed from physics — it follows directly from the uniqueness of n determined by $a^2 r^2 = 2/n^2$.*

13 The Tower of Degenerate Levels

For fixed b , the complete tower of degenerate orbits is:

Level n	Radius r_n	Parameter a_n	Dim. eigenspace
0	∞	0	1
1	$\sqrt{b}$	$\sqrt{2}/\sqrt{b}$	2
2	$\sqrt{b}/2$	$2\sqrt{2}/\sqrt{b}$	2
3	$\sqrt{b}/3$	$3\sqrt{2}/\sqrt{b}$	2
n	$\sqrt{b}/n$	$n\sqrt{2}/\sqrt{b}$	2

Theorem 9 (Absolute invariant of the tower). *The product $a_n r_n$ is constant across all levels:*

$$a_n \cdot r_n = \frac{n\sqrt{2}}{\sqrt{b}} \cdot \frac{\sqrt{b}}{n} = \sqrt{2}$$

$$\boxed{a_n r_n = \sqrt{2} \quad \forall n \geq 1}$$

This invariant is independent of n , b , x_0 , and f . It is an absolute invariant of $\mathcal{O}$.

Remark 9 (The constant $\sqrt{2}$). *The value $\sqrt{2}$ is not free. It comes from the coefficient 2 in $\partial^2 z / \partial a^2 = 2Q/a^3$, which itself comes from the orbital equation. The same 2 appears in the continuous eigenvalue $\lambda = 2/a^2$, in the quantisation $a^2 r^2 = 2/n^2$, and in the degeneracy condition $a^2 b = 2$. The number $\sqrt{2}$ is intrinsic to $\mathcal{O}$.*

14 Asymptotic Behaviour of Eigenfunctions

The eigenfunction at level n is $f_n^{(R,\phi)}(\theta) = e^{R \cos(n\theta - \phi)}$. The parameter $R \geq 0$ controls the concentration.

Lemma 12 (Two regimes). *As R varies, the eigenfunction passes through two qualitatively distinct regimes:*

- $R \rightarrow 0$: $f_n^{(R,\phi)} \rightarrow 1$, the constant function. Only the $k = 0$ Bessel coefficient survives: $I_0(0) = 1$, $I_k(0) = 0$ for $k \neq 0$. The eigenfunction is flat — one mode.
- $R \rightarrow \infty$: all Bessel coefficients grow at the same rate $I_k(R) \sim e^R / \sqrt{2\pi R}$, independently of k . The eigenfunction concentrates on the point $\theta = \phi/n$ — all modes active, equally weighted.

$$R \rightarrow 0 : \text{flat, one mode} \quad R \rightarrow \infty : \text{concentrated, all modes}$$

Lemma 13 (Norm and boundary). *The norm of $f_n^{(R,\phi)}$ in $H_{\mathcal{O}}$ is:*

$$\|f_n^{(R,\phi)}\|_{H_{\mathcal{O}}}^2 = \frac{1}{2\pi} \int_0^{2\pi} R^2 \cos^2(n\theta - \phi) d\theta = \frac{R^2}{2}$$

The amplitude R is the radial coordinate of $H_{\mathcal{O}}$: the norm is linear in R .

As $R \rightarrow \infty$, the norm diverges. Eigenfunctions with $R = \infty$ lie outside $H_{\mathcal{O}}$ — they are boundary objects.

Lemma 14 (Spectral boundary is S^1). *As $R \rightarrow \infty$, the normalised eigenfunction converges in the sense of distributions to a Dirac mass:*

$$\frac{f_n^{(R,\phi)}}{\|f_n^{(R,\phi)}\|_{H_{\mathcal{O}}}} \rightarrow \delta\left(\theta - \frac{\phi}{n}\right)$$

The spectral boundary of $H_{\mathcal{O}}$ is the circle S^1 parametrised by $\phi \in [0, 2\pi)$.

Theorem 10 (Two-component boundary of $\bar{\mathcal{O}}$). *The completion $\bar{\mathcal{O}}$ has a boundary with two components of distinct nature:*

$$\partial\bar{\mathcal{O}} = \underbrace{\{x_0 : P(x_0) = 0\}}_{\substack{\text{radial boundary} \\ \text{discrete, algebraic}}} \cup \underbrace{S^1}_{\substack{\text{spectral boundary} \\ \text{continuous, analytic}}}$$

The radial boundary comes from the Hausdorff completion (orbits shrinking to centres). The spectral boundary comes from $H_{\mathcal{O}}$ (eigenfunctions concentrating to points). Both were constructed independently. Both converge to the same completed space $\bar{\mathcal{O}}$.

15 The Spectral Measure of $\Delta_{\mathcal{O}}$

We determine the measure μ on $\text{Spec}(\Delta_{\mathcal{O}})$ such that for all $f \in H_{\mathcal{O}}$:

$$\|f\|_{H_{\mathcal{O}}}^2 = \int \lambda d\mu(\lambda)$$

Discrete contribution

The spectral decomposition of $\ln f$ in $H_{\mathcal{O}}$:

$$\ln f(\theta) = \sum_{n=0}^{+\infty} R_n \cos(n\theta - \phi_n)$$

Each level n contributes $R_n^2/2$ to the norm. The discrete spectral measure is a mass at $\lambda = b$:

$$\mu_{\text{disc}} = \left(\sum_{n=0}^{+\infty} \frac{R_n^2}{2} \right) \delta(\lambda - b) = \frac{1}{2} \|\ln f\|_{L^2}^2 \cdot \delta(\lambda - b)$$

Continuous contribution

The continuous spectrum $\lambda = 2/a^2$ is parametrised by $a \in \mathbb{R}^*$. The natural measure on $\mathbb{R}^*$ is the Haar measure $da/|a|$. Under the change of variables $\lambda = 2/a^2$:

$$\frac{da}{|a|} = \frac{1}{\sqrt{2}} \lambda^{-3/2} d\lambda$$

Lemma 15 (Spectral density). *The continuous spectral measure is absolutely continuous with density:*

$$\rho(\lambda) = \frac{C}{\lambda^{3/2}}, \quad \lambda > 0$$

where C is a normalisation constant. The density diverges as $\lambda \rightarrow 0^+$: large- a orbits (small eigenvalue) contribute with infinite weight.

Complete spectral measure

Theorem 11 (Spectral measure of $\Delta_{\mathcal{O}}$). *The spectral measure of $\Delta_{\mathcal{O}}$ on $H_{\mathcal{O}}$ is:*

$$d\mu = \underbrace{\frac{C}{\lambda^{3/2}} d\lambda}_{\text{continuous on } (0, +\infty)} + \underbrace{\frac{1}{2} \|\ln f\|_{L^2}^2 \cdot \delta(\lambda - b)}_{\text{discrete at } \lambda=b}$$

This is the Lebesgue–Radon–Nikodym decomposition of μ : absolutely continuous part plus singular (atomic) part.

At the degeneracy condition $a^2b = 2$, the supports of both parts coincide at $\lambda = b$. The total measure at this point is mixed: a finite density plus a point mass.

Remark 10 (Mixed spectrum). *A self-adjoint operator with mixed spectral measure — neither purely continuous nor purely discrete — is the signature of operators with competing structural scales. In $\mathcal{O}$, the two scales are a (governing the continuous part) and b (governing the discrete part). They are independent parameters. Their independence produces the mixed spectrum.*

Summary of New Results

Result	Statement	Status
Quantisation	$a^2 r^2 = 2/n^2, n \in \mathbb{Z}^*$	Established
Discrete eigenvalue	$\lambda_n = b$	Established
Complete spectrum	$\{2/a^2\} \cup \{b\}$	Established
Spectral degeneracy	$a^2 b = 2$	Established
Atomic structure	$r_n = \sqrt{b}/n$	Established
Inter-level distance	$d_H = \sqrt{b}/n(n+1)$	Established
Incompleteness	$\mathcal{O}$ not complete	Established
Radial boundary	$\partial_r \mathcal{O} = \text{roots}(P)$	Established
b -flow on levels	Differential shift	Established
a -flow on levels	Homothety	Established
x_0 -flow on levels	Preservation	Established
Harmonic structure	$r_n/r_m = m/n$, absolute	Established
Three-level topology	Structural, functional, spectral	Established
Eigenfunctions	$f_n^{(R,\phi)} = e^{R \cos(n\theta - \phi)}$	Established
Bessel coefficients	$I_k(R)$ as transition coefficients	Established
Hilbert space $H_{\mathcal{O}}$	Logarithmic inner product	Established
Self-adjointness	$\Delta_{\mathcal{O}}$ on $H_{\mathcal{O}}$	Established
Completeness	Product expansion of admissible f	Established
Multiplicity	Level $n \geq 1$: dim 2; level 0: dim 1	Established
Exclusion principle	At most one degenerate level per orbit	Established
Tower invariant	$a_n r_n = \sqrt{2}$, absolute	Established
Two regimes	$R \rightarrow 0$: flat; $R \rightarrow \infty$: concentrated	Established
Spectral boundary	$\partial_s \mathcal{O} = S^1$	Established
Two-component boundary	Discrete (roots of P) $\cup$ continuous (S^1)	Established
Spectral density	$\rho(\lambda) = C/\lambda^{3/2}$	Established
Spectral measure	Mixed: ac part + atomic part	Established
Radon–Nikodym	Decomposition at degeneracy point	Established
Normalisation C	Explicit constant in $\rho(\lambda)$	Open
Spectral flow	How μ varies with (a, b)	Open

Acknowledgements

This work is the direct continuation of *Orbital Functional Geometry* (2026). The open directions of that paper became the theorems of this one. The orbital Hilbert space $H_{\mathcal{O}}$, the Bessel structure of eigenfunctions, the exclusion principle, and the tower invariant $a_n r_n = \sqrt{2}$ all emerged from following the mathematics without imposing structure.

The author thanks Claude (Anthropic) for sustained mathematical dialogue throughout the development of this work.

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